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Optical analogue of the dynamics of the atom

H G Venkatesh and J Ram

Department of Physics, Birla Institute of Technology, Mesra, Ranchi-835215, India

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Abstract. The mathematical structures of the theories of a two-level atom interacting with radiation and an electromagnetic wave interacting with a dielectric medium in which it is propagated as a plane wave are identical. The spin structure of the Jones operator in optics, representing the polarizer, is obtained in its general form in terms of Stokes parameters, and identified with the optical density operator with a spin structure identical with the density operator in the quantum mechanics of the atom.

While the quantum dynamical equation of the atom can be reduced to the gyroscopic form, the correspondence law giving expression to the parallelism of the two processes leads to forms of the dynamical law in optics which are identical with those of the quantum theory of the atom. In this way one obtains the Hamiltonian H_L of the photon. H_L is a function of angles which characterize the state of polarization in its graphical representation as an ellipse and play the role of generalized coordinates. It leads at once, through Hamilton's canonical equations of classical mechanics, to the spin angular momentum and the connection between the spin and polarization.

Dynamics and polarization are two aspects of the same phenomenon, namely, an interaction process which leads essentially to phase shifts and rotations, in absorptive as in scattering processes. The equation of the photon can be interpreted in terms of propagation in a dielectric medium or in terms of interaction with the two-level atom equally well. In view of the separate equations of the photon and the atom in mutual interaction, of identical form, one can regard the absorptive process as a collision process involving action and reaction.

1. Introduction

An elementary system is said to be polarized if one can assign a vector to it. The electron is polarized because of its spin angular momentum. Electromagnetic radiation is polarized since a vector can be assigned to it, namely the Stokes vector, which also characterizes the state of polarization. An atom with a pair of isolated levels and making transitions between them under the influence of radiation is also polarized in this sense as Feynman *et al* (1957) showed.

But Feynman *et al* did not emphasize the polarization aspect and considered only the dynamical aspect of the theory. As a matter of fact the basic idea of a relation between polarization and dynamics was already present in the equation (Dirac 1958):

$$\frac{l-im}{1-n} = \frac{1+n}{l+im} = \frac{a}{b} \quad (1.1)$$

where l , m , n are the direction cosines of the spin vector σ (of unit length) whose eigenstate is a superposition of the eigenstates of σ_z and a , b are the probability

coefficients of these eigenstates. This result only needs to be taken a little further to obtain the parameters l , m , n in terms of transition probability coefficients:

$$\begin{aligned} l &= ab^* + ba^* \\ m &= +i(ab^* - ba^*) \\ n &= (aa^* - bb^*). \end{aligned} \tag{1.2}$$

These direction cosines are none other than the parameters introduced by Feynman *et al* in the dynamics of the atom. The work of Feynman *et al* had two merits: it treated the more general case of a two-level system which includes the atom and the electron as special cases and it showed that the gyroscopic equation in atomic dynamics follows directly from Schrödinger's wave equation.

A general theorem can be stated: that polarization and dynamics are two aspects, or two alternative modes of description, of an interaction process. One may look for the gyroscopic equation in the field equations of the electromagnetic wave process in a dielectric medium and easily find it. The dynamical equation of polarization thus provides the basis for a direct comparison of the electromagnetic wave process in a dielectric medium and the dynamics of an atom under the influence of radiation.

It is not merely that a vector can be assigned to the atom but that in the work of Feynman *et al* its components are formally identical in structure with the Stokes parameters in optics. This is more than one would readily expect. Further, the angular momentum operators of the radiation field in quantum electrodynamics are identical in structure with the Stokes parameters of polarization and hence also with the atomic parameters. This leads one to explore more fully the direct connection between polarization of the classical wave field and the spin of the photon on the one hand and, on the other, the analogy between the parametric state of the electromagnetic field in the dielectric medium and that of the atom. The analogue of the atom in a state of transitions is the electromagnetic wave in the dielectric and the analogue of the field interacting with the atom is the dielectric! The two processes being compared are the absorption (or emission) of radiation by an atom, considered as a two-level system, and the propagation of an electromagnetic wave through a dielectric medium whose effects include double refraction and optical activity. There are, of course, marked differences apart from the fact that the atom is a quantal system and the wave traversing the dielectric a classical system. The atom is losing energy to the interacting field (or gaining energy from it) but there is no such energy exchange between the dielectric medium and the electromagnetic wave. Rather there is exchange of energy between the two states of polarization as the wave traverses the medium.

There is an extensive literature on the subject of interaction between fermions and photons involving the formalism of polarization. However, we feel that the essence of the matter has not been touched on and that the fundamental significance of the identity of the formal structures of atomic dynamics and the electromagnetic wave process to the theory of radiation itself has not been explored. This identity leads directly to the quantization of the radiation field and to the basic equation of the photon in interaction with matter. This paper is mainly concerned with showing the parallelism between the two processes in §§ 2 and 3 and, in § 4, the implication of this parallelism to the extension of classical optics to quantum optics.

To distinguish the term polarization used here from other types (magnetic, dielectric) it will be referred to as optical polarization, although our considerations are not limited to the visible range of frequencies. The field inducing transitions in the atom will

be referred to as the interacting field and the electromagnetic wave propagated in the dielectric medium will be referred to as the optical wave to avoid possible confusion.

2. Polarization and dynamics: the atomic case

2.1. Kinematical structure

The state of a two-level system interacting with radiation is represented as a superposition of the eigenstates:

$$\Psi = a\Psi_1 + b\Psi_2 \quad (2.1)$$

where Ψ_1 and Ψ_2 are the eigenstates to which the energy values $E_1, E_2 (E_1 > E_2)$ belong, the transition frequency ω_0 being given by

$$E_1 - E_2 = \hbar\omega_0$$

(\hbar is Planck's constant divided by 2π).

In matrix form the state is written as follows:

$$\mathbf{V} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\mathbf{V}_1 + b\mathbf{V}_2. \quad (2.2)$$

The normalization of the wavefunction is expressed by

$$aa^* + bb^* = 1. \quad (2.3)$$

The state Ψ can be obtained from a unitary transformation

$$\mathbf{T} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \quad (2.4a)$$

of the initial state which is represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ if the atom is initially in the state of excitation and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ if it is initially in the ground state:

$$\Psi = \mathbf{T}\mathbf{V}_1 \quad \text{or} \quad \Psi = \mathbf{T}\mathbf{V}_2. \quad (2.4b)$$

The density operator for the two-level system (Venkatesh and Dixit 1970) is given by

$$\rho = \frac{1}{2}(r_0\sigma_0 + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3) \quad (2.5)$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli operators, $\sigma_0 = \mathbf{1}$ and

$$\begin{aligned} r_1 &= ab^* + a^*b \\ r_2 &= i(ab^* - a^*b) \end{aligned} \quad (2.6)$$

$$\begin{aligned} r_3 &= aa^* - bb^* \\ r_0 &= aa^* + bb^*. \end{aligned} \quad (2.7)$$

The normalizing condition on \mathbf{r} is expressed by

$$r_1^2 + r_2^2 + r_3^2 = r_0^2 = 1.$$

r_1, r_2, r_3 can be regarded as the components of a unit vector \mathbf{r} . In matrix form this vector can be represented by

$$\mathbf{r} = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 = 2\rho - \mathbf{1}. \quad (2.8)$$

The initial state of \mathbf{r} is given by $\pm\sigma_3$. The plus sign applies to the state of excitation and the minus sign to the ground state. \mathbf{r} results from transformation of σ_3 by \mathbf{T} :

$$\mathbf{r} = \mathbf{T}\sigma_3\bar{\mathbf{T}}. \quad (2.9)$$

A direct connection between the wavefunction Ψ and the parameters r_1, r_2, r_3 can be established by noting

$$\rho\Psi = \Psi.$$

Using the vector representation of Ψ and the normalization condition, this leads to

$$\frac{a}{b} = z = \frac{r_1 - ir_2}{1 - r_3} = \frac{1 + r_3}{r_1 + ir_2} \quad (2.10)$$

which is exactly the same as (1.1) although it applies more generally to two-level systems and is analogous to the Poincaré representation of polarization in optics (Venkatesh and Sarkar 1976). (2.10) can be resolved into two expressions with an arbitrary phase constant δ :

$$a = \frac{1}{\sqrt{2}} \frac{r_1 - ir_2}{(1 - r_3)^{1/2}} e^{i\delta}$$

$$b = \frac{1}{\sqrt{2}} (1 - r_3)^{1/2} e^{i\delta}.$$

The vector \mathbf{r} can be represented in terms of angles:

$$\begin{aligned} r_1 &= \sin \theta \sin \phi \\ r_2 &= \sin \theta \cos \phi \\ r_3 &= \cos \theta. \end{aligned} \quad (2.11)$$

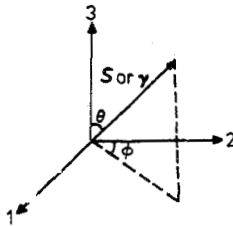


Figure 1. The Σ coordinate system.

Thus,

$$z = a/b = -i \cot(\frac{1}{2}\theta) e^{i\phi}. \quad (2.12)$$

Putting $\delta = \frac{1}{2}(\psi - \phi + \pi)$, which implies no further restriction on δ since the angle ψ is arbitrary,

$$\begin{aligned} a &= \cos(\frac{1}{2}\theta) e^{\frac{1}{2}i(\phi+\psi)} \\ b &= i \sin(\frac{1}{2}\theta) e^{-\frac{1}{2}i(\phi-\psi)} \end{aligned} \quad (2.13)$$

$$\mathbf{T} = \begin{pmatrix} \cos(\frac{1}{2}\theta) e^{\frac{1}{2}i(\phi+\psi)} & i \sin(\frac{1}{2}\theta) e^{\frac{1}{2}i(\phi-\psi)} \\ i \sin(\frac{1}{2}\theta) e^{-\frac{1}{2}i(\phi-\psi)} & \cos(\frac{1}{2}\theta) e^{-\frac{1}{2}i(\phi+\psi)} \end{pmatrix}. \quad (2.14)$$

This is a product of three rotations $\mathbf{T}(\theta)$, $\mathbf{T}(\phi)$, $\mathbf{T}(\psi)$ (Euler matrices).

2.2. Dynamical law

We shall consider the interaction of a radiation field at the position of an atom and regard the atom as having no permanent dipole moment in the eigenstates. The spin structure of the Hamiltonian is given by (Venkatesh and Dixit 1970):

$$H = \frac{1}{2}\hbar \mathbf{w} \cdot \boldsymbol{\sigma} = \frac{1}{2}\hbar(w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3) \quad (2.15)$$

where

$$\begin{aligned} w_1 &= -\dot{\theta} \cos \omega t \\ w_2 &= -\dot{\theta} \sin \omega t \\ w_3 &= w_0. \end{aligned} \quad (2.16)$$

θ is the pulse strength (apart from a factor γ/\hbar) and is given by

$$\theta = \frac{\gamma E_0}{2\hbar} t. \quad (2.17)$$

γ is the transition dipole moment, $\frac{1}{2}E_0$ the amplitude of the radiation applied in the 1-2 plane and ω its angular frequency. $w_{\pm} = w_1 \pm iw_2 = -\dot{\theta} e^{\pm i\omega t}$ represents the interaction of the radiation with the atom and is a product of two factors: $\dot{\theta}$ giving the coupling between radiation and the atom, and the exponential involving the radiation frequency.

The dynamics of the atom is obtained from the equation of motion in quantum mechanics in any of its forms. It is most easily obtained from the dynamical law:

$$i\hbar \partial T / \partial t = HT \quad (2.18)$$

by first solving the problem in the rotating coordinate system (Venkatesh and Roy 1971). The dynamical law determines the meaning of θ and ϕ in the T matrix (2.14). Under conditions of resonance ($\omega = w_0$) the solution is given by (2.14) with $\theta = (\gamma E_0 / 2\hbar)t$, $\phi = w_0 t$. \mathbf{r} is given by (2.11) with the signs \pm, \mp, \mp for r_1, r_2, r_3 corresponding to the initial states (of lower and higher energy respectively).

Although the quantum mechanical problem is thereby completely solved it is of particular interest here to exploit other forms of the dynamical law. They are

$$i\hbar \partial \rho / \partial t = -(\rho H - H \rho) \quad (2.19)$$

$$d\mathbf{r} / dt = \mathbf{w} \times \mathbf{r}. \quad (2.20)$$

The latter follows from (2.19) and (2.5) and from the identity

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) - (\mathbf{b} \cdot \boldsymbol{\sigma})(\mathbf{a} \cdot \boldsymbol{\sigma}) = 2i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}.$$

The density operator is directly related to the polarization aspect of the two-level system and to the gyroscopic equation which describes the motion of the polarization vector. This motion is a representative of the atomic transition.

3. Polarization and dynamics: the optical case

The kinematical structure of optical polarization is exactly similar to that of the atomic case. To distinguish the two physical cases one might replace \mathbf{a}, \mathbf{b} by c_1, c_2 . In the optical case, however, one has

$$c_1 = \frac{1}{2}\mathcal{E}_1, \quad c_2 = \frac{1}{2}\mathcal{E}_2 e^{i\alpha} \quad (3.1)$$

where $\mathcal{E}_1, \mathcal{E}_2$ are the amplitudes and α the phase difference of the vibrations along two mutually perpendicular axes. The matrices in (2.14) have their optical analogues as the interaction matrices which transform the state of polarization as the wave progresses through the dielectric.

The dynamical variables of optical theory are expressible, as in the two-level atomic case, in the general form:

$$\alpha = \sum_{\mu} a_{\mu} \sigma_{\mu} \quad (\mu = 1, 2, 3, 0).$$

Polarizers, retarders and optically active media are devices for the measurement of eigenstates. We shall work out one very simple case to illustrate how the relevant operator for a device can be found. For a linear polarizer with the transmission axis taken as horizontal (direction 1 of a fixed coordinate system) the operator α_1 is defined by

$$\alpha_1 \mathbf{V}_1 = \mathbf{V}_1 \quad \alpha_1 \mathbf{V}_2 = 0.$$

σ_0 and σ_3 are the only operators which act on the eigenstates in this way and the general form of α can be written as

$$\alpha_1 = \frac{1}{2}(\sigma_0 + \sigma_3)$$

remembering the normalizing condition on \mathbf{V}_1 . In fact the Pauli operators have simple meanings in optical theory. σ_1 represents a half-wave retarder with the fast axis at an angle $\pm\pi/4$, σ_2 a half-wave circular retarder and σ_3 a linear half-wave retarder with the fast axis at angle 0° . σ_- and σ_+ are excitation and de-excitation operators effecting transitions $\mathbf{V}_1 \rightarrow \mathbf{V}_2, \mathbf{V}_2 \rightarrow \mathbf{V}_1$ respectively.

The general form of the eigenvalue equation $\alpha_1 \mathbf{V}_1 = \mathbf{V}_1$ is obtained by the unitary transformation

$$\mathbf{T} \mathbf{V}_1 = \mathbf{V}, \quad \alpha_1^* = \alpha = \mathbf{T} \alpha_1 \bar{\mathbf{T}}.$$

On the left-hand side we have $\mathbf{T} \alpha_1 \bar{\mathbf{T}} \mathbf{T} \mathbf{V}_1 = (\mathbf{T} \alpha_1 \bar{\mathbf{T}}) \mathbf{T} \mathbf{V}_1$. Hence

$$\alpha \mathbf{V} = \mathbf{V} \tag{3.2a}$$

where

$$\alpha = \frac{1}{2}(s_0 \sigma_0 + s_1 \sigma_1 + s_2 \sigma_2 + s_3 \sigma_3). \tag{3.2b}$$

$s_{\mu} (\mu = 1, 2, 3, 0)$ are given by expressions similar to (2.6) and (2.7). s_1, s_2, s_3 are the Stokes parameters characterizing the state of polarization.

The general expression for α gives the dynamical variable α (relevant to the polarizer) in terms of Stokes parameters and relates the state of polarization of the incident beam to the direction of the transmission axis of the polarizer required for full transmission.

The eigenvalue equation (3.2a) is useful in expressing the state \mathbf{V} in terms of Stokes parameters. The procedure is the same as in the atomic case and leads to exactly similar results which may here be omitted.

The ratio of the coefficients c_1/c_2 can now be obtained in terms of s_1, s_2, s_3 :

$$\frac{c_1}{c_2} = z = \frac{s_1 - i s_2}{1 - s_3}. \tag{3.3}$$

This is analogous to the expression for the ratio of probability amplitudes a, b in the atomic case. s_1, s_2, s_3 correspond to the direction cosines of the spin direction in the electron case mentioned in § 1. Thus the eigenvalue equation (3.2a) brings out the important relationship of optical theory, namely that between polarization and dynamics when the physical parameters $\mathcal{E}_1, \mathcal{E}_2, \alpha$ are allowed to vary with time. The representation of \mathbf{S} in terms of c_1, c_2 thus occupies a central position in optical theory.

\mathbf{S} may be represented in polarization space in the coordinate system Σ (figure 1), exactly as in (2.11). The relation between the Stokes parameters and the angles of the graphical representation of polarization as an ellipse (Jauch and Rohrlich 1955) is

$$\tan(2\gamma) = \frac{s_2}{(s_1^2 + s_3^2)^{1/2}}, \quad \tan(2\delta) = \frac{s_1}{s_3}$$

where δ is the azimuthal angle and γ the ellipticity angle. In the Σ representation

$$\tan(2\gamma) = \frac{\sin \theta \cos \phi}{1 - \sin^2 \theta \cos^2 \phi}, \quad \tan 2\delta = \tan \theta \sin \phi.$$

There is another representation of \mathbf{S} in terms of angles in Σ' illustrated in figure 2:

$$s_1 = \cos \theta' \sin \phi'$$

$$s_2 = \sin \theta'$$

$$s_3 = \cos \theta' \cos \phi'.$$

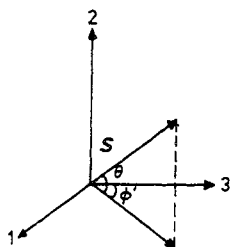


Figure 2. The Σ' coordinate system.

The first representation gives a very simple connection between the physical parameters $\mathcal{E}_1, \mathcal{E}_2, \alpha$ and the angles θ, ϕ :

$$\mathcal{E}_1 = 2 \cos(\frac{1}{2}\theta), \quad \mathcal{E}_2 = 2 \sin(\frac{1}{2}\theta),$$

$$\tan \alpha = s_2/s_1 = \cot \phi \quad \text{or} \quad \phi = -(\alpha + \frac{1}{2}\pi).$$

The field amplitudes depend on θ only and the phase difference on ϕ only. The second representation gives a direct connection between the angle θ', ϕ' and the angles appearing in the graphical representation of polarization (see figure 3):

$$\phi' = 2\delta \tag{3.4}$$

$$\theta' = 2\gamma.$$

Twice the azimuthal angle is the longitude ϕ' and twice the ellipticity angle is the latitude θ' of the unit sphere with 2-axis taken as the polar axis (see figure 3).

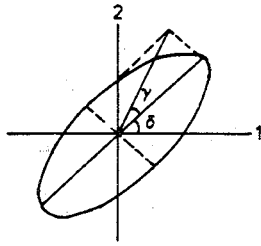


Figure 3. Graphical representation of polarization.

3.1. Optical density operator

The operator α is not only the general form of the operator representing a polarizer which is completely transparent to light whose state of polarization is specified by S , it is also the optical density operator from which intensity functions can be obtained in classical optics. Given a beam A whose state of polarization is specified by S , the corresponding polarizer operator is $\alpha = \frac{1}{2}(1 + S \cdot \sigma)$ for a device which is completely transparent to the beam A in state V :

$$\alpha V = V.$$

Let us consider a second beam B in state $V' = \begin{pmatrix} a \\ b \end{pmatrix}$, the corresponding parametric state being S' and the polarizer operator

$$\alpha' = \frac{1}{2}(1 + S' \cdot \sigma)$$

referring to a device which is completely transparent to B. Note that $\alpha' V' = V'$. What is the intensity of the beam A after transmission through the second device? An operator acting on the state V will, in general, change the state V which may be represented as the superposition of orthonormal states V' and V'' . For a polarizer, say α' ,

$$\alpha' V = p V'$$

(from the meaning of a polarizer). Thus

$$p = V' \alpha' V = V' V = aa'^* + bb'^*.$$

The intensity of the beam B in state V' contained in the beam A in state V is therefore

$$I' = pp^* = (aa'^* + bb'^*)(a^* a' + b^* b')$$

or

$$I' = \frac{1}{2}(1 + S \cdot S').$$

The results are symmetrical with respect to A, α and B, α' . On the other hand let us form the trace of the product $\alpha \alpha'$:

$$\begin{aligned} \text{Tr}(\alpha \alpha') &= \frac{1}{4} \text{Tr}[(1 + S \cdot \sigma)(1 + S' \cdot \sigma)] \\ &= \frac{1}{4} \text{Tr}[(1 + S \cdot \sigma + S' \cdot \sigma + S \cdot \sigma S' \cdot \sigma)] = \frac{1}{2}(1 + S \cdot S'). \end{aligned}$$

Thus the $\text{Tr}(\alpha \alpha')$ gives the intensity of the beam B after transmission through the first device which is completely transparent to beam A. The symmetry of this expression with respect to beams A and B is obvious. In other words α enables us to calculate the probability of selecting from a given beam in a definite state of polarization S , a beam B

in any arbitrarily chosen state of polarization S' . Hence α has the significance of the optical density operator. We may express this explicitly by

$$\alpha = \rho_L = \frac{1}{2}(S_0\sigma_0 + \mathbf{S} \cdot \boldsymbol{\sigma}). \quad (3.5)$$

This is essentially the coherence matrix for monochromatic waves (Born and Wolf 1970).

Generally the dynamical variables of two-level systems have the following form:

$$\beta = p + q(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)$$

where p, q are c numbers like a . However, the a are not normalized. If $N = (a_1^2 + a_2^2 + a_3^2)^{1/2}$ is the normalizing constant, β can be written as

$$\beta = p + Nq\alpha$$

where $\alpha = \frac{1}{2}(S' \cdot \boldsymbol{\sigma}')$ and $S' = \mathbf{a} / N$. The expectation value of β is then

$$\langle \beta \rangle = p + q\mathbf{S} \cdot \mathbf{a}. \quad (3.6)$$

3.2. Dynamical law

The Stokes vector which specifies the state of polarization of an electromagnetic wave is so far considered to be static. Its general motion is a gyroscopic motion with $\mathcal{E}_1, \mathcal{E}_2, \alpha$ varying with time when the wave is propagated in a dielectric medium. For an optically active medium, the interaction angular velocities w_1, w_2, w_3 can be determined from the $D-E$ relation involving the gyration vector $\boldsymbol{\gamma}$. The interaction velocities w_1, w_2, w_3 are again given by (2.16) in terms of θ and ϕ where $\phi = -(\alpha + \frac{1}{2}\pi)$, α being the phase difference. For a progressive wave in a doubly refractive medium

$$\alpha = \frac{2\pi c}{\lambda}(t - t_0) \left(\frac{1}{n_1} - \frac{1}{n_2} \right) \quad (3.7)$$

where n_1, n_2 are the refractive indices for the fields E_1, E_2 in the medium and $(t - t_0)$ is the time of flow of the energy of the optical wave into the dielectric medium.

$$\dot{\alpha} = w_3 = w \left(\frac{1}{n_1} - \frac{1}{n_2} \right). \quad (3.8)$$

The full symmetry between the atomic and optical cases is established by allowing the amplitudes $\mathcal{E}_1, \mathcal{E}_2$ to vary with time subject to

$$s_0 = \frac{1}{4}(\mathcal{E}_1^2 + \mathcal{E}_2^2) = \frac{1}{2}I_0 = \text{constant}$$

$$\dot{\mathcal{E}}_1 = -\frac{1}{2}\mathcal{E}_2\dot{\theta} \quad \dot{\mathcal{E}}_2 = \frac{1}{2}\mathcal{E}_1\dot{\theta}.$$

From the normalizing condition ($I_0 = 2, S_0 = 1$)

$$\dot{\mathcal{E}}_1^2 + \dot{\mathcal{E}}_2^2 = \dot{\theta}^2.$$

The state of polarization which in general is elliptical at any instant in time varies with time because of the time dependence of α . The optical rotatory effect is further superposed on the time-varying elliptical polarization. The angle θ for an optically active medium is simply given by

$$\theta = 2\delta = w(n'_1 - n'_2)t \quad (3.9)$$

where n'_1, n'_2 are the refractive indices for the right and left circular beams of light.

3.3. Equations of motion in optics

Since \mathbf{w} and \mathbf{S} can be represented by two-dimensional matrices

$$\begin{aligned}\mathbf{S} \rightarrow \mathbf{S} &= S_1 \sigma_1 + S_2 \sigma_2 + S_3 \sigma_3 = 2\rho_L - 1 \\ \mathbf{w} \rightarrow \mathbf{w} &= w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3\end{aligned}$$

we can, with the aid of the identity (3.6), write the gyroscopic equation in matrix form by simply scalar multiplying it by σ :

$$i\dot{\rho}_L = \frac{1}{2}(\mathbf{w}\rho_L - \rho_L\mathbf{w}). \quad (3.10)$$

In terms of the T operator the equation of motion takes the form:

$$i\dot{\mathbf{T}} = \frac{1}{2}\mathbf{w}\mathbf{T}. \quad (3.11)$$

In terms of the state, the dynamical law can also be obtained from (2.13), now valid for c_1, c_2 :

$$i\dot{\mathbf{V}} = \frac{1}{2}\mathbf{w}\mathbf{V}. \quad (3.12)$$

The procedure essentially is as follows:

$$\mathbf{w}\mathbf{V} = \mathbf{w}(\mathbf{T}\mathbf{V}_1) = (\mathbf{w}\mathbf{T})\mathbf{V}_1 = (2i\dot{\mathbf{T}})\mathbf{V}_1 = 2i(\dot{\mathbf{T}}\mathbf{V}_1) = 2i\dot{\mathbf{V}}.$$

4. Quantization of the optical wave

We note that in the classical optics and quantum mechanics of the two-level atom, the states, the parameters, the transformation operators and the density operators are identical in structure and physical significance. We may therefore regard optical polarization and the dynamics of the atom as representatives of an interaction process between a two-level system and a force field. We may raise the status of this parallelism to a correspondence law which enables one to transform the equations of motion in optics ((3.10), (3.11) and (3.12)) into forms of the dynamical law for the atom by multiplying them through by \hbar and thus obtain the equations of quantum optics:

$$i\hbar\dot{\rho}_L = -(\rho_L H_L - H_L \rho_L); \quad \rho_L = \frac{1}{2}(1 + \mathbf{S} \cdot \boldsymbol{\sigma}) \quad (4.1)$$

$$i\hbar\dot{\mathbf{T}} = H_L \mathbf{T} \quad (4.2)$$

$$i\hbar\dot{\mathbf{V}} = H_L \mathbf{V} \quad (4.3)$$

provided we interpret

$$H_L = \frac{1}{2}\hbar\mathbf{w} = \frac{1}{2}\hbar(w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3) \quad (4.4)$$

as the energy operator of the optical field. It is a simple linear function of the interaction velocity components w_1, w_2, w_3 . H_L in optics has the same structure as in atomic theory. The interaction can be interpreted either in terms of the dielectric medium interacting with light or in terms of the radiation interacting with the atom. Thus, in optics

$$w_1 = w(n'_1 - n'_2) \sin \alpha, \quad w_2 = -w(n'_1 - n'_2) \cos \alpha, \quad w_3 = w\left(\frac{1}{n_1} - \frac{1}{n_2}\right). \quad (4.5)$$

Or, in atomic mechanics,

$$w_1 = -\frac{\gamma E_0}{2\hbar} \cos \phi, \quad w_2 = +\frac{\gamma E_0}{2\hbar} \sin \phi, \quad w_3 = w_0. \quad (4.6)$$

(Note the relation $\phi = -(\alpha + \frac{1}{2}\pi)$ and the correspondence $w(n'_1 - n'_2)t \rightarrow (\gamma E_0/2\hbar)t$ (equations (2.17) and (3.9)).) In the latter case the equation of motion is perfectly symmetrical with respect to the photon and the two-level atom.

We have in (3.5) the optical density operator of a classical wave with the state of polarization given by \mathcal{S} . It is also the density operator of a single polarized photon in quantum theory. It is a simple linear function of the Stokes parameters and may be taken over into quantum field theory if c_1, c_2 are regarded as boson operators (Mehta and Sharma 1974).

4.1. The Hamiltonian of the optical wave

(4.4) gives the quantum mechanical Hamiltonian. The classical Hamiltonian function is the expectation value of the energy operator:

$$\langle H_L \rangle = \text{Tr}(\rho_L H_L). \quad (4.7)$$

From the general formula (3.6)

$$\langle H_L \rangle = \frac{1}{2}\hbar(w_1 s_1 + w_2 s_2 + w_3 s_3). \quad (4.8)$$

This is essentially the classical energy function which appears as the scalar product of the interaction vector \mathbf{w} and the state vector \mathcal{S} . Let us investigate the structure of this Hamiltonian.

4.1.1. Energy. From the angle representation of \mathbf{w} and \mathcal{S} in Σ :

$$w_1 s_1 + w_2 s_2 = 0 \quad (4.9)$$

so that the expectation value of the Hamiltonian in optics is

$$\langle H_L \rangle = \frac{1}{2}\hbar w_3 s_3 \quad (4.10)$$

which is identical with the expression of Feynman *et al* (1957) for the case of the atom. In order to find the energy of the optical wave *in vacuo* which is polarized along the 1- or 2-direction we consider a polarizer (see (3.8)) for which

$$\begin{aligned} n_1 = 1, n_2 = \infty, \frac{1}{2}\pi + \alpha = -\phi = wt; & \quad w_1 = w_2 = 0, w_3 = \dot{\alpha} = -\dot{\phi} = w \\ n_1 = \infty, n_2 = 1, \frac{1}{2}\pi + \alpha = -\phi = wt; & \quad w_1 = w_2 = 0, w_3 = \dot{\alpha} = -\dot{\phi} = -w. \end{aligned}$$

In either case, corresponding to the states of polarization $s_3 = \pm 1$

$$\langle H_L \rangle = \frac{1}{2}\hbar w. \quad (4.11)$$

The quantity on the left-hand side is just half the total energy of the radiation field in volume V representing the electrical part of it only, so that we may put for the total energy

$$E = h\nu. \quad (4.12)$$

We conclude that corresponding to a single atom in a state of transition, the representative optical wave contains *one* quantum of light of energy $h\nu$. This conclusion would not

be unique however, unless there was additional support from atomic mechanics itself; namely the interpretation of $aa^* = \sin^2(\theta/2)$ and $bb^* = \cos^2(\theta/2)$ as probabilities for the photon. The quantization of the wave field implying the appearance of the individual photon necessarily introduces probabilistic concepts. This is the first important consequence of the correspondence law. The linear momentum follows from the classical relation $\mathbf{p} = \mathbf{S}/c$ where \mathbf{S} is the Poynting vector:

$$p = h\nu/c. \quad (4.13)$$

4.1.2. Spin angular momentum. From the relation between the angles of the Σ' representation and those of the graphical representation (3.4), the Hamiltonian operator (4.4) is expressed as

$$H_L = \hbar(\dot{\gamma}_1\sigma_1 + \dot{\delta}\sigma_2 + \dot{\gamma}_3\sigma_3). \quad (4.14)$$

Note that in this representation σ_2 is diagonal and

$$\begin{aligned} \mathbf{w} &\rightarrow \begin{pmatrix} w'_2 & w'_1 - iw'_3 \\ w'_1 + iw'_3 & -w'_2 \end{pmatrix}, & \mathbf{r} &\rightarrow \begin{pmatrix} r'_2 & r'_1 - ir'_3 \\ r'_1 + ir'_3 & -r'_2 \end{pmatrix} \\ w'_1 = \dot{\theta}'_1 = -\dot{\theta}' \cos \phi', & w'_2 = \dot{\phi}', & w'_3 = \dot{\theta}'_2 = \dot{\theta}' \sin \phi' \end{aligned} \quad (4.15)$$

and $\dot{\gamma}_1 = \dot{\gamma} \cos \phi'$, $\dot{\gamma}_3 = \dot{\gamma} \sin \phi'$ analogously to (4.15).

Regarding the angles γ_1 , δ , γ_3 as generalized canonical coordinates, we obtain the conjugate canonical momenta from Hamilton's equations:

$$\partial H_L / \partial \dot{\gamma}_1 = \hbar \sigma_1, \quad \partial H_L / \partial \dot{\delta} = \hbar \sigma_2, \quad \partial H_L / \partial \dot{\gamma}_3 = \hbar \sigma_3. \quad (4.16)$$

Their commutation rules follow from those of the Pauli operators:

$$M_K = \hbar \sigma_K \quad (4.17)$$

$$M_1 M_2 - M_2 M_1 = 2i\hbar M_3 \quad \text{etc.} \quad (4.18)$$

The eigenvalues of the spin angular momentum follow from the second equation in (4.17):

$$M_2 = \pm \hbar. \quad (4.19)$$

Note that there are no angles in physical space corresponding to the velocities $\dot{\gamma}_1$, $\dot{\gamma}_3$ which are merely formal. Hence M_1 , M_3 cannot be regarded as spin components.

The connection between spin and polarization parameters is established as follows. In Σ' representation the classical Hamiltonian is

$$\langle H_L \rangle = \hbar(\dot{\gamma}_1 s_1 + \dot{\delta} s_2 + \dot{\gamma}_3 s_3) \quad (4.20)$$

and

$$\partial \langle H_L \rangle / \partial \dot{\gamma}_1 = \hbar s_1, \quad \partial \langle H_L \rangle / \partial \dot{\delta} = \hbar s_2, \quad \partial \langle H_L \rangle / \partial \dot{\gamma}_3 = \hbar s_3. \quad (4.21)$$

The angular momentum is thus

$$M_2 = \hbar s_2. \quad (4.22)$$

For circularly polarized light ($s_2 = \pm 1$) this leads to the eigenvalues (4.19). Right circularly polarized light can be assigned the value $+\hbar$ and left circularly polarized light the value $-\hbar$. This is the second important consequence of the correspondence law. In view of the relation

$$w'_1 s_1 + w'_3 s_3 = 0$$

the expression for the energy becomes

$$\langle H_L \rangle = \pm \hbar \dot{\delta}$$

which is a product of the angular momentum of the photon and the angular velocity and is similar to the energy-momentum relation $E = pc$ for the photon. This derivation is similar to the derivation of the magnetic moment of the electron by obtaining the interaction energy in an external magnetic field. Consistently with the probabilistic interpretation one concludes that for linear light ($s_1 = 1; s_3 = 1$), which is a superposition of right and left circularly polarized beams of light, the angular momentum is zero since the spins are oppositely directed with equal probability.

Similar remarks to those following (4.19) can be made. This is perhaps connected with the lack of gauge invariance of the components M_1, M_3 regarded as functions of the boson operators. Nevertheless one could introduce them to obtain the commutation rules (4.18) to find matrix representation of M_2 . Note, however, that (4.22) is now a classical definition (in spite of \hbar !) and turns out to be the spin operators in quantum field theory.

4.2. The electron case

The results of the previous analysis should be contrasted with those for the electron for which the canonical variables are θ, ϕ and

$$H = \frac{1}{2}\hbar(\dot{\theta}_1\sigma_1 + \dot{\theta}_2\sigma_2 + \dot{\phi}\sigma_3). \quad (4.23)$$

The canonical variables are realised already as angles in physical space characterizing the parametric state of the electron. Thus for the electron

$$M_K = \frac{1}{2}\hbar\sigma_K \quad (4.24)$$

and the eigenvalues are

$$M_3 = \pm \frac{1}{2}\hbar. \quad (4.25)$$

The difference between the photon and the two-level material system should therefore be traced to the generalized coordinates in the two cases. The correspondence law has to do with the formal structure but does not extend to the *values* of the dynamical variables. Analogously to (4.21) for the photon we have

$$\partial(H)/\partial\dot{\theta}_1 = \frac{1}{2}\hbar r_1, \quad \partial(H)/\partial\dot{\theta}_2 = \frac{1}{2}\hbar r_2, \quad \partial(H)/\partial\dot{\phi} = \frac{1}{2}\hbar r_3 \quad (4.26)$$

giving the connection between the spin and the parameters r_1, r_2, r_3 . It is on the strict basis provided by equations (4.21) and (4.26) that the correspondence between the states of polarization of the classical wave and the direction of the spin of the fermion can be established.

4.3. Details of the analogy between optics and atomic transition theory

The same results follow from an examination of the corresponding relationships between optical polarization theory and atomic dynamics; this examination is not without interest. From (3.7)

$$w_3 = w_0 \rightarrow w \left(\frac{1}{n_1} - \frac{1}{n_2} \right)$$

or

$$\hbar\omega_0 = E_1 - E_2 \rightarrow \hbar\omega \left(\frac{1}{n_1} - \frac{1}{n_2} \right).$$

We note that the expectation value of the Hamiltonian of the atom is

$$\langle H_A \rangle = \frac{1}{2} \hbar\omega_0 r_3.$$

This can be recast as follows in resonant interaction:

$$\langle H_A \rangle = \frac{1}{2} \hbar\omega_0 \cos \theta = \frac{1}{2} \hbar\omega_0 [\cos^2(\theta/2) - \sin^2(\theta/2)]$$

or

$$\langle H_A \rangle = E_1 \cos^2(\theta/2) + E_2 \sin^2(\theta/2).$$

When we transform this into its optical analogue it is necessary to introduce the common scale factor for dimensional reasons and put (§ 3)

$$\cos^2(\theta/2) \rightarrow A \mathcal{E}_1^2 / 4$$

$$\sin^2(\theta/2) \rightarrow A \mathcal{E}_2^2 / 4$$

so that we may express the correspondence between optics and atomic mechanics as:

$$E_1 \rightarrow \hbar\omega/n_1 \quad E_2 \rightarrow \hbar\omega/n_2. \quad (4.27)$$

Thus

$$\langle H_A \rangle \rightarrow A \hbar\omega \left(\frac{1}{n_1} \frac{\mathcal{E}_1^2}{4} + \frac{1}{n_2} \frac{\mathcal{E}_2^2}{4} \right). \quad (4.28)$$

On the other hand the same quantity, regarded as the energy flowing across area F in t seconds, is given by electromagnetic theory (see (4.32)):

$$S't = \frac{V}{4\pi} \left(\frac{1}{n_1} \frac{\mathcal{E}_1^2}{4} + \frac{1}{n_2} \frac{\mathcal{E}_2^2}{4} \right). \quad (4.29)$$

On comparing (4.28) and (4.29)

$$A = V/4\pi\hbar\omega.$$

After introducing this dimensional term the components are:

Quantity in volume V	$S' = \text{flux across area } F = S/t$
$r_1 \rightarrow s_1 = \frac{V}{4\pi\hbar\omega} \mathcal{E}_1 \mathcal{E}_2 \cos \alpha$	$s'_1 = \frac{c}{4\pi\hbar\omega} F \mathcal{E}_1 \mathcal{E}_2 \cos \alpha \quad (4.30a)$
$r_2 \rightarrow s_2 = \frac{V}{4\pi\hbar\omega} \mathcal{E}_1 \mathcal{E}_2 \sin \alpha$	$s'_2 = \frac{c}{4\pi\hbar\omega} F \mathcal{E}_1 \mathcal{E}_2 \sin \alpha \quad (4.30b)$
$r_3 \rightarrow s_3 = \frac{V}{8\pi\hbar\omega} (\mathcal{E}_1^2 - \mathcal{E}_2^2)$	$s'_3 = \frac{c}{8\pi\hbar\omega} F (\mathcal{E}_1^2 - \mathcal{E}_2^2) \quad (4.30c)$
$r_0 \rightarrow s_0 = \frac{V}{8\pi\hbar\omega} (\mathcal{E}_1^2 + \mathcal{E}_2^2)$	$s'_0 = \frac{c}{8\pi\hbar\omega} F (\mathcal{E}_1^2 + \mathcal{E}_2^2) \quad (4.30d)$

The three physical quantities E , p , $M_z = M_3$ can also be defined classically for the optical wave (Baym 1969):

energy:

$$W = \frac{n_1^2 \mathcal{E}_1^2 + n_2^2 \mathcal{E}_2^2}{4\pi} V \quad (4.31)$$

linear momentum:

$$\mathbf{p} = \frac{\mathbf{S}}{c} F; \quad \mathbf{S} = \frac{c}{4\pi} \left(\frac{1}{n_1} \mathcal{E}_1^2 + \frac{1}{n_2} \mathcal{E}_2^2 \right) \mathbf{n}_1 \quad (4.32)$$

angular momentum along z :

$$M_z = \frac{V}{4\pi w} \mathcal{E}_1 \mathcal{E}_2 \sin \alpha \quad (4.33)$$

where \mathbf{S} is the Poynting vector directed along the unit vector \mathbf{n}_1 (along z), w the angular frequency of the wave, V the volume, F the area of cross section of flow of energy and $V = Fct$.

Since the time average of the total energy W is given by

$$W = \frac{\mathcal{E}_1^2 + \mathcal{E}_2^2}{8\pi} V$$

we have from (4.30d)

$$W = h\nu$$

since $s_0 = 1$. We are therefore dealing throughout with only *one* photon in the field as already remarked.

From (4.30b) and (4.33)

$$M_z = \hbar s_2.$$

At the same time the atom in the transitional state would acquire, by the correspondence law, two properties: namely a pseudo-linear momentum of magnitude $\frac{1}{2}\hbar w_0 r_3/c$ and angular momentum $M_z = \frac{1}{2}\hbar r_2$ both varying with time.

5. Summary and discussion

The formalism of polarization in classical optics is identical with the quantum mechanical formalism except for the commutation rule underlying the philosophy of quantum theory. This apparently new parallelism can be traced to the isomorphism of $SU(2)$ and O_{+3} and the fact that the two physical systems are two-level systems whose dynamics can be expressed in terms of gyroscopic motion. The ultimate significance of this parallelism is that the optical field is quantizable and that the dynamics of the atom and of the photon in mutual interaction can be put on a symmetrical footing. If the atom introduces phase shifts and rotations (or in other words changes the state of polarization of the photon) the photon effects similar changes in the atom. The absorptive (or emissive) process is therefore a collision process involving two particles. In view of the relationship between dynamics and polarization, the continuous change in the intensity of the photon beam interacting with the atom implies a continuous variation of s_3 and

can be regarded as due to the flow of energy from one state of polarization ($s_3 = +1$) to another ($s_3 = -1$).

The parallelism means that the structure of the Hamiltonian for the photon is identical with that of the atom. The quantal properties of the electromagnetic wave immediately follow from this Hamiltonian. The graphical representation of polarization, far from being merely a vivid picture of the polarization in elementary optics, plays its proper role in quantum theory by providing generalized coordinates, namely the azimuthal angle and the ellipticity angle, for generating angular momenta. Thereby, also, the picture of polarization in optics becomes very precise and identical with that of the electron. The Stokes parameters acquire a new meaning; s_2 , s_3 are regarded as two independent variables giving the angular momentum and energy, and are associated with a *single* photon. The connection between spin and polarization becomes clear.

Thus optics gains an important dimension from dynamics. (This is similar in spirit to the method of quantizing the particle motion in Schrödinger's original method by establishing the analogy between optics and classical mechanics.) One could ascribe new properties to the optical process, the fields \mathcal{E}_1 , \mathcal{E}_2 being interpreted in terms of probability amplitudes a , b and the refractive indices n_1 , n_2 characterizing the two eigenstates of the wave process in the refractive medium.

On the other hand, the theory of atomic transition also gains. One can ascribe a phase ($\phi = \omega t$) and state of polarization $r(\theta, \phi)$ to a two-level system. The free atom is linearly polarized in the eigenstates, horizontally if it is in the state of higher energy and vertically if it is in the ground state. Its more general state of polarization corresponds to a superposed state just after interaction with a radiation field for a limited duration of time. In general, during the interaction, the state of polarization is continuously changing with time as determined by the gyroscopic equation.

Of course the analogy can say nothing about the *values* of any dynamical variables or the nature of the particles. The photon is a boson and the electron (or the two-level system) a fermion. This difference can be traced to the existence of types of generalized coordinates for the two cases. For the atom they are the measurable angles θ , ϕ of the spin representation, and for the photon they are $\phi' = 2\delta$, $\theta' = 2\gamma$ of the graphical representation (the ellipse).

One could go further in the use of the analogy to obtain an elegant solution of the atomic transition problem in terms of the Poincaré representation of optical polarization by stereographic projection of the point of the unit sphere on to the complex plane (Venkatesh and Sarkar 1976). The parallelism could be extended to almost every physical situation in quantum mechanics which is analogous to an optical context.

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